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Transitivity in coherence-based probability logic ☆

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A R T I C L E   I N F O

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A B S T R A C T

We study probabilistically informative (weak) versions of transitivity by using suitable definitions of defaults and negated defaults in the setting of coherence and imprecise probabilities. We represent p-consistent sequences of defaults and/or negated defaults by g-coherent imprecise probability assessments on the respective sequences of conditional events. Moreover, we prove the coherent probability propagation rules for Weak Transitivity and the validity of selected inference patterns by proving p-entailment of the associated knowledge bases. Finally, we apply our results to study selected probabilistic versions of classical categorical syllogisms and construct a new version of the square of opposition in terms of defaults and negated defaults.

1. Motivation and outline

While Transitivity is basic for (monotonic) reasoning, it does not hold in nonmonotonic reasoning systems (e.g., [30]). Therefore, various patterns of Weak Transitivity were studied in the literature (e.g., [19]). In probabilistic approaches, Transitivity is probabilistically non-informative, i.e., the probabilities of the premises $p(C|B)$ and $p(B|A)$ do not constrain the probability of the conclusion $p(C|A)$ (for instance, the

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extension $p(C|A) = z$ of the assessment $p(C|B) = 1, p(B|A) = 1$ is coherent for any $z \in [0, 1]$; see [41,42]). In this paper, we study probabilistically informative versions of Transitivity in the setting of coherence [4,14,24,26]. Transitivity has also been studied in [7,17]; among other differences, in our approach we use imprecise probabilities in the setting of coherence, where conditioning events may have zero probability.

After introducing some notions of coherence for set-valued probability assessments (Section 2), we present probabilistic interpretations of defaults and negated defaults (Section 3). We represent a knowledge base by a sequence of defaults and/or negated defaults, which we interpret by an imprecise probability assessment on the associated sequence of conditional events. Moreover, we generalize definitions of p-consistency and p-entailment. In Section 4 we prove the coherent probability propagation rules for Weak Transitivity (Theorem 3 and Theorem 4). We then exploit Theorem 3 to demonstrate the validity of selected patterns of (weak) transitive inferences involving defaults and negated defaults by proving p-entailment of the corresponding knowledge bases (Section 5). Finally, we illustrate how our results can be applied to investigate classical categorical syllogisms (Section 6) and to analyze the traditional square of opposition (Section 7) within coherence-based probability logic.

2. Imprecise probability assessments

Given two events E and H, with $H \neq \bot$, the *conditional event* $E|H$ is defined as a three-valued logical entity which is *true* if EH (i.e., $E \wedge H$) is true, *false* if $\neg EH$ is true, and *void* if H is false. Given a finite sequence of $n \geq 1$ conditional events $\mathcal{F} = (E_1|H_1, \ldots, E_n|H_n)$, we denote by $\mathcal{P}$ *any precise* probability assessment $\mathcal{P} = (p_1, \ldots, p_n)$ on $\mathcal{F}$, where $p_j = p(E_j|H_j) \in [0, 1]$, $j = 1, \ldots, n$. Moreover, we denote by Π the set of *all coherent precise* assessments on $\mathcal{F}$. The coherence-based approach to probability has been adopted by many authors (see, e.g., [4,6,9,14,20,27,28,36–38,41,43]); we therefore recall only selected key features of coherence in this paper. We recall that when there are no logical relations among the events $E_1, H_1, \ldots, E_n, H_n$ involved in $\mathcal{F}$, that is $E_1, H_1, \ldots, E_n, H_n$ are logically independent, then the set Π associated with $\mathcal{F}$ is the whole unit hypercube $[0, 1]^n$. If there are logical relations, then the set Π *could be* a strict subset of $[0, 1]^n$. As it is well known $\Pi \neq \emptyset$; therefore, $\emptyset \neq \Pi \subseteq [0, 1]^n$.

Definition 1. An *imprecise, or set-valued, assessment* $\mathcal{I}$ on a family of conditional events $\mathcal{F}$ is a (possibly empty) set of precise assessments $\mathcal{P}$ on $\mathcal{F}$.

Definition 1, introduced in [21], states that an *imprecise (probability) assessment* $\mathcal{I}$ on a given family $\mathcal{F}$ of n conditional events is just a (possibly empty) subset of $[0, 1]^n$. Given an imprecise assessment $\mathcal{I}$ we denote by $\mathcal{I}^c$ the *complementary imprecise assessment* of $\mathcal{I}$, i.e. $\mathcal{I}^c = [0, 1]^n \setminus \mathcal{I}$. In what follows, we generalize the notions of g-coherence, coherence, and total-coherence for interval-valued probability assessments (see, e.g., [24, Definitions 7a, 7b, 7c, respectively]) to the case of imprecise (in the sense of set-valued) probability assessments.

Definition 2. Let a sequence of n conditional events $\mathcal{F}$ be given. An imprecise assessment $\mathcal{I} \subseteq [0, 1]^n$ on $\mathcal{F}$ is *g-coherent* if and only if there exists a coherent precise assessment $\mathcal{P}$ on $\mathcal{F}$ such that $\mathcal{P} \in \mathcal{I}$.

Definition 3. Let $\mathcal{I}$ be a subset of $[0, 1]^n$. For each $j \in \{1, 2, \ldots, n\}$, the projection $\rho_j(\mathcal{I})$ of $\mathcal{I}$ onto the j-th coordinate, is defined as

$$\rho_j(\mathcal{I}) = \{x_j \in [0, 1] : p_j = x_j, \text{ for some } (p_1, \ldots, p_n) \in \mathcal{I}\}.$$

Definition 4. An imprecise assessment $\mathcal{I}$ on a sequence of n conditionals events $\mathcal{F}$ is *coherent* if and only if, for every $j \in \{1, \ldots, n\}$ and for every $x_j \in \rho_j(\mathcal{I})$, there exists a coherent precise assessment $\mathcal{P} = (p_1, \ldots, p_n)$ on $\mathcal{F}$, such that $\mathcal{P} \in \mathcal{I}$ and $p_j = x_j$.

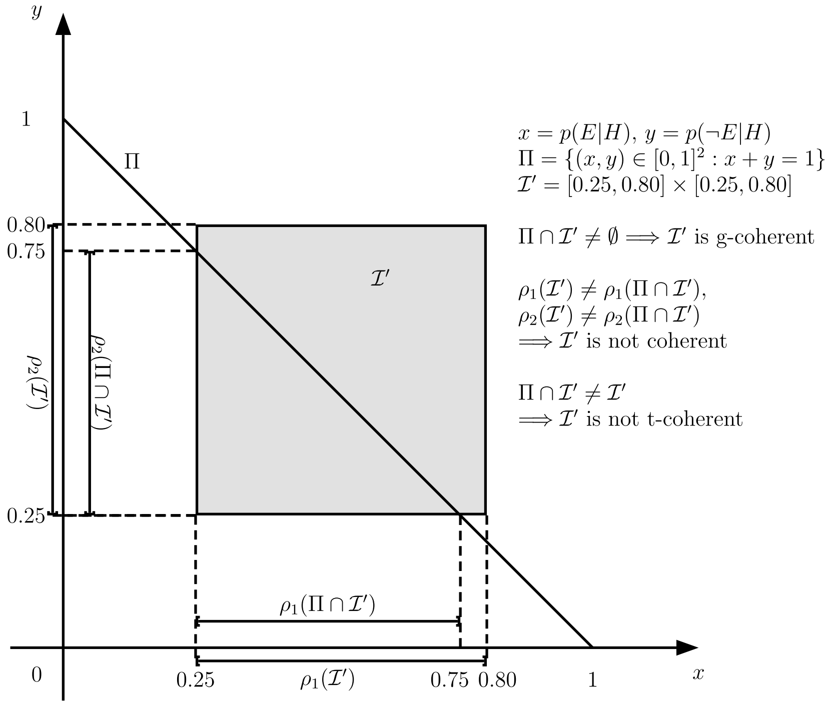

Fig. 1. The g-coherent assessment $\mathcal{I}'$ on $(E|H, \neg E|H)$ explained in Example 1.

Definition 5. An imprecise assessment $\mathcal{I}$ on $\mathcal{F}$ is *totally coherent* (t-coherent) if and only if the following two conditions are satisfied: (i) $\mathcal{I}$ is non-empty; (ii) if $\mathcal{P} \in \mathcal{I}$, then $\mathcal{P}$ is a coherent precise assessment on $\mathcal{F}$.

Remark 1. We observe that:

$$\begin{aligned}
&\mathcal{I} \text{ is g-coherent } &&\Longleftrightarrow \Pi \cap \mathcal{I} \neq \emptyset \Longleftrightarrow \forall j \in \{1, \dots, n\},\ \rho_j(\Pi \cap \mathcal{I}) \neq \emptyset\,;\\
&\mathcal{I} \text{ is coherent } &&\Longleftrightarrow \forall j \in \{1, \dots, n\},\ \emptyset \neq \rho_j(\Pi \cap \mathcal{I}) = \rho_j(\mathcal{I})\,;\\
&\mathcal{I} \text{ is t-coherent } &&\Longleftrightarrow \emptyset \neq \Pi \cap \mathcal{I} = \mathcal{I}\,.
\end{aligned}$$

Then, the following relations among the different notions of *coherence* hold: $\mathcal{I}$ is t-coherent $\Rightarrow \mathcal{I}$ is coherent $\Rightarrow \mathcal{I}$ is g-coherent.

In the following example we illustrate the different notions of coherence.

Example 1. Given two logically independent events E and H, with $H \neq \bot$, the set of all coherent precise assessments on the pair $\mathcal{F} = (E|H, \neg E|H)$ is obviously the segment $\Pi = \{(x, 1-x), x \in [0, 1]\}$. Let us consider three imprecise assessments on $\mathcal{F}$ which, we will see, differ with respect to the three notions of coherence: $\mathcal{I}' = [0.25, 0.80] \times [0.25, 0.80]$; $\mathcal{I}'' = [0.25, 0.75] \times [0.25, 0.75]$; $\mathcal{I}''' = \{(x, 1-x) : x \in [0.25, 0.75]\}$. The assessment $\mathcal{I}'$ is g-coherent because $\Pi \cap \mathcal{I}'$ is the (non-empty) segment with extreme points $(0.25, 0.75), (0.75, 0.25)$, see Fig. 1; we also observe that $\mathcal{I}'$ is not coherent and not t-coherent. The assessment $\mathcal{I}''$ is coherent because $\emptyset \neq \rho_1(\Pi \cap \mathcal{I}'') = [0.25, 0.75] = \rho_1(\mathcal{I}'')$ and $\emptyset \neq \rho_2(\Pi \cap \mathcal{I}'') = [0.25, 0.75] = \rho_2(\mathcal{I}'')$ (see Fig. 2); we notice that $\mathcal{I}''$ is g-coherent but not t-coherent. The assessment $\mathcal{I}'''$ is t-coherent because $\emptyset \neq \Pi \cap \mathcal{I}''' = \mathcal{I}'''$ (see Fig. 3); of course $\mathcal{I}'''$ is coherent and g-coherent as well. Finally, we note that any subset $\mathcal{I}$ of $[0, 1]^2$ such that $\Pi \cap \mathcal{I} = \emptyset$ is not g-coherent, not coherent, and not t-coherent.

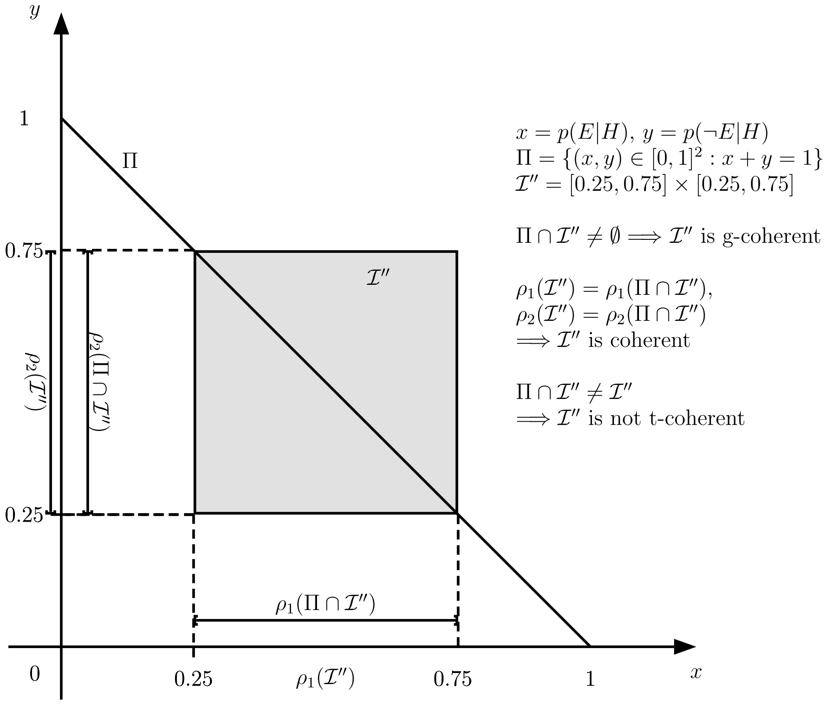

Fig. 2. The coherent assessment $\mathcal{I}''$ on $(E|H, \neg E|H)$ explained in Example 1.

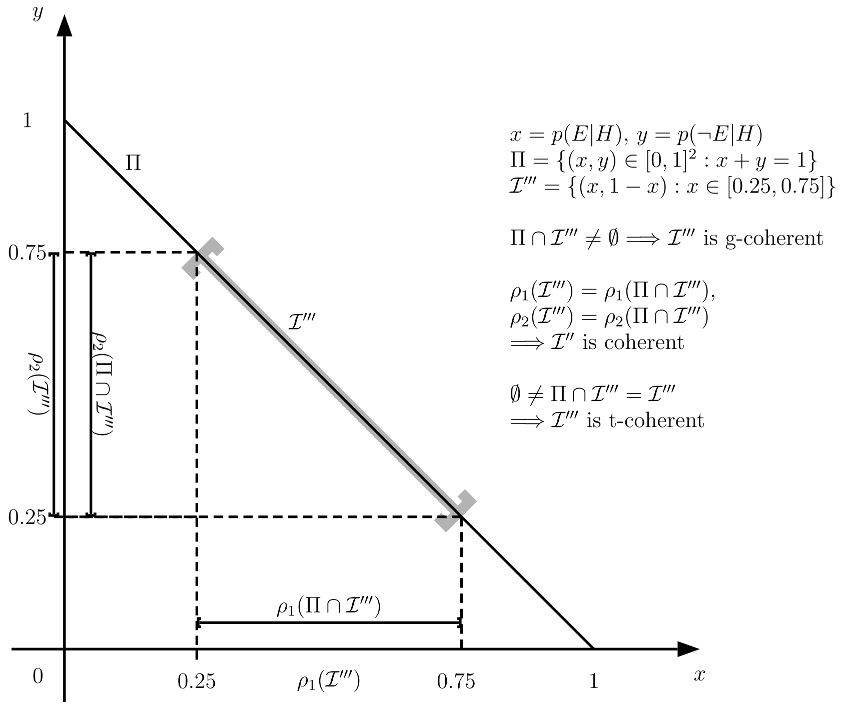

Fig. 3. The t-coherent assessment $\mathcal{I}'''$ on $(E|H, \neg E|H)$ explained in Example 1.

Definition 6. Let $\mathcal{I}$ be a non-empty subset of $[0, 1]^n$. For each sub-vector $(j_1, \dots, j_m)$ of $(1, \dots, n)$, the projection $\rho_{(j_1,\dots,j_m)}(\mathcal{I})$ of $\mathcal{I}$ onto the coordinates $(j_1, \dots, j_m)$, with $1 \leq m \leq n$, is defined as the set $\rho_{(j_1,\dots,j_m)}(\mathcal{I}) \subseteq [0, 1]^m$ such that each vector $(x_{j_1}, \dots, x_{j_m}) \in \rho_{(j_1,\dots,j_m)}(\mathcal{I})$ is the sub-vector $(p_{j_1}, \dots, p_{j_m})$ of some $\mathcal{P} = (p_1, \dots, p_n) \in \mathcal{I}$.

Let $\mathcal{I}$ be an imprecise assessment on the sequence $\mathcal{F} = (E_1|H_1, \dots, E_n|H_n)$; moreover, let $E_{n+1}|H_{n+1}$ be a further conditional event and let $\mathcal{J} \subseteq [0,1]^{n+1}$ be an imprecise assessment on $(\mathcal{F}, E_{n+1}|H_{n+1})$. We say that $\mathcal{J}$ is an *extension* of $\mathcal{I}$ to $(\mathcal{F}, E_{n+1}|H_{n+1})$ iff $\rho_{(1,\dots,n)}(\mathcal{J}) = \mathcal{I}$, that is: (*i*) for every $(p_1, \dots, p_n, p_{n+1}) \in \mathcal{J}$, it holds that $(p_1, \dots, p_n) \in \mathcal{I}$; (*ii*) for every $(p_1, \dots, p_n) \in \mathcal{I}$, there exists $p_{n+1} \in [0,1]$ such that $(p_1, \dots, p_n, p_{n+1}) \in \mathcal{J}$.

Definition 7. Let $\mathcal{I}$ be a g-coherent assessment on $\mathcal{F} = (E_1|H_1, \dots, E_n|H_n)$; moreover, let $E_{n+1}|H_{n+1}$ be a further conditional event and let $\mathcal{J}$ be an extension of $\mathcal{I}$ to $(\mathcal{F}, E_{n+1}|H_{n+1})$. We say that $\mathcal{J}$ is a *g-coherent extension* of $\mathcal{I}$ if and only if $\mathcal{J}$ is g-coherent.

Theorem 1. *Given a g-coherent assessment $\mathcal{I} \subseteq [0,1]^n$ on $\mathcal{F}$, let $E_{n+1}|H_{n+1}$ be a further conditional event. Then, there exists a g-coherent extension $\mathcal{J} \subseteq [0,1]^{n+1}$ of $\mathcal{I}$ to the family $(\mathcal{F}, E_{n+1}|H_{n+1})$.*

Proof. As $\mathcal{I}$ is g-coherent, there exists a coherent precise assessment $\mathcal{P}$ on $\mathcal{F}$, with $\mathcal{P} \in \mathcal{I}$. Then, as it is well known, there exists (a non-empty interval) $[p', p''] \subseteq [0,1]$ such that $(\mathcal{P}, p_{n+1})$ is a coherent precise assessment on $(\mathcal{F}, E_{n+1}|H_{n+1})$, for every $p_{n+1} \in [p', p'']$ (Fundamental Theorem of Probability; see, e.g., [4,13,16,31]). Now, let any $\Gamma \subseteq [0,1]$ be given such that $\Gamma \cap [p', p''] \neq \emptyset$; moreover, consider the extension $\mathcal{J} = \mathcal{I} \times \Gamma$ on $(\mathcal{F}, E_{n+1}|H_{n+1})$. Clearly, $(\mathcal{P}, p_{n+1}) \in \mathcal{J}$ for every $p_{n+1} \in \Gamma \cap [p', p'']$; moreover the assessment $(\mathcal{P}, p_{n+1})$ on $(\mathcal{F}, E_{n+1}|H_{n+1})$ is coherent for every $p_{n+1} \in \Gamma \cap [p', p'']$. So by Definition 2, $\mathcal{J}$ is a g-coherent extension of $\mathcal{I}$ to $(\mathcal{F}, E_{n+1}|H_{n+1})$. □

Given a g-coherent assessment $\mathcal{I}$ on a sequence of n conditional events $\mathcal{F}$, for each coherent precise assessment $\mathcal{P}$ on $\mathcal{F}$, with $\mathcal{P} \in \mathcal{I}$, we denote by $[\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}]$ the interval of coherent extensions of $\mathcal{P}$ to $E_{n+1}|H_{n+1}$; that is, the assessment $(\mathcal{P}, p_{n+1})$ on $(\mathcal{F}, E_{n+1}|H_{n+1})$ is coherent if and only if $p_{n+1} \in [\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}]$. Then, defining the set

$$\Sigma = \bigcup\nolimits_{\mathcal{P} \in \Pi \cap \mathcal{I}} [\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}] \,, \tag{1}$$

for every $p_{n+1} \in \Sigma$, the assessment $\mathcal{I} \times \{p_{n+1}\}$ is a g-coherent extension of $\mathcal{I}$ to $(\mathcal{F}, E_{n+1}|H_{n+1})$; moreover, for every $p_{n+1} \in [0,1] \setminus \Sigma$, the extension $\mathcal{I} \times \{p_{n+1}\}$ of $\mathcal{I}$ to $(\mathcal{F}, E_{n+1}|H_{n+1})$ is not g-coherent. Thus, denoting by Π' the set of coherent precise assessments on $(\mathcal{F}, E_{n+1}|H_{n+1})$, it holds that Σ is the projection onto the $(n+1)$-th coordinate of the set $(\mathcal{I} \times [0,1]) \cap \Pi'$, that is $\rho_{n+1}((\mathcal{I} \times [0,1]) \cap \Pi') = \Sigma$. We say that Σ is the *set of coherent extensions* of the imprecise assessment $\mathcal{I}$ on $\mathcal{F}$ to the conditional event $E_{n+1}|H_{n+1}$.

3. Probabilistic knowledge bases and entailment

Let E and H denote events, where H is a not self-contradictory event. The sentence "*E is a plausible consequence of H*" is a *default*, which we denote by $H \mid\!\sim E$ (following the notation in [19,30]). Moreover, we denote a *negated default*, $\neg(H \mid\!\sim E)$, by $H \mid\!\not\sim E$ (read: "*it is not the case that: E is a plausible consequence of H*"). We define defaults and negated defaults in terms of probabilistic assessments as follows:

Definition 8. Given two events E and H we say that $H \mid\!\sim E$ (resp., $H \mid\!\not\sim E$) holds iff our imprecise probability assessment $\mathcal{I}$ on $E|H$ is $\mathcal{I} = \{1\}$ (resp., $\mathcal{I} = [0,1[$).

We observe that a default is negated by classical negation: the default $H \mid\!\sim E$ is represented by the assessment $\{1\}$ on $E|H$ and the negated default $H \mid\!\not\sim E$ is represented by the assessment $[0,1[$, which is the complementary set of $\{1\}$. Thus, we require that $\neg(H \mid\!\not\sim E) = \neg(\neg(H \mid\!\sim E)) = (H \mid\!\sim E)$. Given two events E and H, with $H \neq \bot$, by coherence $p(E|H) + p(\neg E|H) = 1$ (which holds in general). Thus, the probabilistic interpretation of the following types of sentences $H \mid\!\sim E$, $H \mid\!\sim \neg E$, $H \mid\!\not\sim \neg E$, and $H \mid\!\not\sim E$

Table 1
Probabilistic interpretations of defaults (types A and E) and negated defaults (types I and O), and their respective (imprecise) assessments $\mathcal{I}$ on a conditional event $E|H$.

Type	Sentence	Probabilistic constraint	Assessment $\mathcal{I}$ on $E\vert H$
A	$H \mid\!\sim E$	$p(E\vert H) = 1$	$\{1\}$
E	$H \mid\!\sim \neg E$	$p(\neg E\vert H) = 1$	$\{0\}$
I	$H \not\mid\!\sim \neg E$	$p(\neg E\vert H) < 1$	$]0, 1]$
O	$H \not\mid\!\sim E$	$p(E\vert H) < 1$	$[0, 1[$

can be represented in terms of imprecise assessments on $E|H$ (see Table 1). We recall that the notion of p-consistency for a knowledge base, given by Adams in [1], has been also studied in the framework of coherence (see, e.g., [20]). In [20, Definition 4] Adams' p-consistency of a knowledge base is interpreted by the g-coherence of an imprecise assessment, where $p(E|H) \geq 1 - \varepsilon$ for every $\varepsilon > 0$, i.e. $p(E|H)$ is close to 1, for each default $H \mid\!\sim E$ in the given knowledge base. Therefore, the notion of p-consistency is related to the notion of g-coherence. Moreover, as shown in [26, Definition 2, Remark 1, Theorem 4], p-consistency can be defined equivalently by requiring $p(E|H) = 1$ for each default $H \mid\!\sim E$. Of course, for what concerns practical aspects, instead of the latter approach it is more useful to use imprecise assessments (see, e.g., [20,26,27,39–42]). In this paper a knowledge base $\mathcal{K}$ is defined as a (non-empty) finite sequence of defaults and negated defaults. Let $\mathcal{K} = (H_1 \mid\!\sim E_1, \dots, H_n \mid\!\sim E_n, D_1 \not\mid\!\sim C_1, \dots, D_m \not\mid\!\sim C_m)$ be a knowledge base, with $n + m \geq 1$. We now define our probabilistic representation of the knowledge base $\mathcal{K}$ by a corresponding pair $(\mathcal{F}_\mathcal{K}, \mathcal{I}_\mathcal{K})$, where $\mathcal{F}_\mathcal{K}$ is the ordered family of conditional events $(E_1|H_1, \dots, E_n|H_n, C_1|D_1, \dots, C_m|D_m)$ and $\mathcal{I}_\mathcal{K}$ is the imprecise assessment $\times_{i=1}^{n}\{1\} \times \times_{j=1}^{m}[0, 1[$ on $\mathcal{F}_\mathcal{K}$. We now define the notion of p-consistency of a given knowledge base in terms of g-coherence.

Definition 9. A knowledge base $\mathcal{K}$ is *p-consistent* if and only if the imprecise assessment $\mathcal{I}_\mathcal{K}$ on $\mathcal{F}_\mathcal{K}$ is g-coherent.

In other words, $\mathcal{K} = (H_1 \mid\!\sim E_1, \dots, H_n \mid\!\sim E_n, D_1 \not\mid\!\sim C_1, \dots, D_m \not\mid\!\sim C_m)$ is p-consistent if and only if there exists a coherent precise assessment $\mathcal{P} = (p_1, \dots, p_n, q_1, \dots, q_m)$ on $\mathcal{F}_\mathcal{K} = (E_1|H_1, \dots, E_n|H_n, C_1|D_1, \dots, C_m|D_m)$ such that $p_i = 1$, $i = 1, \dots, n$, and $q_i < 1$, $i = 1, \dots, m$.

Example 2. Let $H \neq \bot$ and Π be the set of all the coherent assessments $x = p(E|H)$. We distinguish three cases. (i) $H \wedge E = \bot$: $\Pi = \{0\}$, $(H \mid\!\sim E)$ is not p-consistent because the assessment $p(E|H) = 1$ is not coherent; $(H \not\mid\!\sim E)$ is p-consistent because the assessment $p(E|H) = 0$ is coherent, hence there exists a coherent assessment $p(E|H)$ such that $p(E|H) < 1$; (ii) $H \wedge \neg E = \bot$: $\Pi = \{1\}$, therefore by the same reasoning, $(H \mid\!\sim E)$ is p-consistent, while $(H \not\mid\!\sim E)$ is not p-consistent; (iii) $H \wedge E \neq \bot$ and $H \wedge \neg E \neq \bot$: $\Pi = [0, 1]$, $(H \mid\!\sim E)$ and $(H \not\mid\!\sim E)$ are separately p-consistent.

We define the notion of p-entailment of a (negated) default from a p-consistent knowledge base in terms of coherent extension of a g-coherent assessment.

Definition 10. Let $\mathcal{K}$ be p-consistent. $\mathcal{K}$ *p-entails* $A \mid\!\sim B$ (resp., $A \not\mid\!\sim B$), denoted by $\mathcal{K} \models_p A \mid\!\sim B$ (resp., $\mathcal{K} \models_p A \not\mid\!\sim B$), iff the (non-empty) set of coherent extensions to $B|A$ of $\mathcal{I}_\mathcal{K}$ on $\mathcal{F}_\mathcal{K}$ is $\{1\}$ (resp., a subset of $[0, 1[$).

Remark 2. We observe that, trivially, for any p-consistent $A \mid\!\sim B$: $A \mid\!\sim B \models_p A \not\mid\!\sim \neg B$. Then, a conclusion of the form $A \mid\!\sim B$ can easily be weakened to $A \not\mid\!\sim \neg B$, i.e., if $\mathcal{K} \models_p A \mid\!\sim B$, then $\mathcal{K} \models_p A \not\mid\!\sim \neg B$.

Theorem 2. *Let $\mathcal{K}$ be p-consistent. $\mathcal{K} \models_p A \mid\!\sim B$ (resp., $\mathcal{K} \models_p A \not\mid\!\sim B$), iff there exists a (non-empty) sub-sequence $\mathcal{S}$ of $\mathcal{K}$: $\mathcal{S} \models_p A \mid\!\sim B$ (resp., $\mathcal{S} \models_p A \not\mid\!\sim B$).*

Proof. ($\Rightarrow$) Trivially, by setting $\mathcal{S} = \mathcal{K}$.

($\Leftarrow$) Assume that $\mathcal{S} \models_p A \mid\!\sim B$ (resp., $A \not\mid\!\sim B$). Then, for every precise coherent assessment $\mathcal{P} \in \mathcal{I}_{\mathcal{S}}$ on $\mathcal{F}_{\mathcal{S}}$, if the extension $(\mathcal{P}, z)$ on $(\mathcal{F}_{\mathcal{S}}, B|A)$ is coherent, then $z = 1$ (resp., $z \neq 1$). Let $\mathcal{P}' \in \mathcal{I}_{\mathcal{K}}$ be a coherent precise assessment on $\mathcal{F}_{\mathcal{K}}$. For *reductio ad absurdum* we assume that the extension $(\mathcal{P}', z)$ on $(\mathcal{F}_{\mathcal{K}}, B|A)$ is coherent with $z \in [0, 1[$ (resp., $z = 1$). Then, the sub-assessment $(\mathcal{P}, z)$ of $(\mathcal{P}', z)$ on $(\mathcal{F}_{\mathcal{S}}, B|A)$ is coherent with $z \in [0, 1[$ (resp., $z = 1$): this contradicts $\mathcal{S} \models_p A \mid\!\sim B$ (resp., $\mathcal{S} \models_p A \not\mid\!\sim B$). Therefore, $\mathcal{K} \models_p A \mid\!\sim B$ (resp., $\mathcal{K} \models_p A \not\mid\!\sim B$). $\square$

A similar approach has been developed in [14, Definition 26] (see also [15]). We observe that if the knowledge base $\mathcal{K}$ consists of defaults only, then Definitions 9 and 10 coincide with the notion of p-consistency and p-entailment, respectively, investigated from a coherence perspective in [26] (see also [5,23,25,27]). Moreover, p-entailment of the inference rules of the well-known nonmonotonic System P has been studied in this context (e.g., [14,20], see also [3,12,18]).

Remark 3. By Table 1 the probabilistic interpretation of $\mathcal{K} = (H_1 \mid\!\sim E_1, \dots, H_n \mid\!\sim E_n, D_1 \not\mid\!\sim C_1, \dots, D_m \not\mid\!\sim C_m)$ can equivalently be represented by the assessment $\mathcal{I}_{\mathcal{K}} = \times_{i=1}^{n}\{1\} \times \times_{j=1}^{m}]0, 1]$ on $\mathcal{F}_{\mathcal{K}} = (E_1|H_1, \dots, E_n|H_n,\ \neg C_1|D_1, \dots, \neg C_m|D_m)$. Definitions 9 and 10 can be rewritten accordingly.

Example 3. Given three logically independent events A, B, C, with $A \neq \bot$, any assessment $(x, y) \in [0, 1]^2$ on $(C|A, B|A)$ is of course coherent. Furthermore, the extension $z = P(C|AB)$ of (x, y) on $(C|A, B|A)$ is coherent if and only if $z \in [z', z'']$, where [20]

$$z' = \begin{cases} \frac{x+y-1}{y} > 0, & \text{if } x + y > 1, \\ 0, & \text{if } x + y \leq 1, \end{cases} \qquad z'' = \begin{cases} \frac{x}{y} < 1, & \text{if } x < y, \\ 1, & \text{if } x \geq y\,. \end{cases} \tag{2}$$

Then, for $x = 1$ and $y = 1$ we obtain $z' = z'' = 1$, that is (see also [14]):

$$\text{(Cautious Monotonicity)} \qquad (A \mid\!\sim C, A \mid\!\sim B) \models_p AB \mid\!\sim C\,.$$

Moreover, for $x = 1$ and for any $y > 0$ it follows that $z' = z'' = 1$; then, we obtain (see also [19]):

$$\text{(Rational Monotonicity)} \qquad (A \mid\!\sim C, A \not\mid\!\sim \neg B) \models_p AB \mid\!\sim C\,. \tag{3}$$

We observe that Rational Monotonicity can be equivalently formulated as follows [14,20,32]:

$$(AB \not\mid\!\sim C, A \not\mid\!\sim \neg B) \models_p A \not\mid\!\sim C\,. \tag{4}$$

Example 4. Given three logically independent events A, B, C, with $A \neq \bot$ and $B \neq \bot$, in [20] it has been proved that any assessment $(x, y) \in [0, 1]^2$ on $(C|A, C|B)$ is coherent. Furthermore, the extension $z = P(C|A \vee B)$ of (x, y) on $(C|A, C|B)$ is coherent if and only if $z \in [z', z'']$, where[4]

$$z' = \begin{cases} \frac{xy}{x+y-xy} > 0, & \text{if } x > 0 \wedge y > 0, \\ 0, & \text{if } x = 0 \vee y = 0, \end{cases} \qquad z'' = \begin{cases} \frac{x+y-2xy}{1-xy} < 1, & \text{if } x < 1 \wedge y < 1, \\ 1, & \text{if } x = 1 \vee y = 1\,. \end{cases}$$

Then, in our framework we obtain (see also [14,20,32]): $(A \mid\!\sim C, B \mid\!\sim C) \models_p A \vee B \mid\!\sim C$ (Or-Rule); $(A \not\mid\!\sim C, B \not\mid\!\sim C) \models_p A \vee B \not\mid\!\sim C$ (Disjunctive Rationality).

[4] Note that $z' = T_0^H(x, y)$ and $z'' = S_0^H(x, y)$, where T_0^H and S_0^H are the Hamacher t-norm and t-conorm (with $\lambda = 0$), respectively (see, e.g., [27]).

4. Weak Transitivity: propagation of probability bounds

In this section, we prove two results on the propagation of a precise, or interval-valued, probability assessment on $(C|B, B|A, A|A \vee B)$ to $C|A$.

Remark 4. Let A, B, C be logically independent events. It can be proved that the assessment (x, y, t) on $\mathcal{F} = (C|B, B|A, A|A \vee B)$ is coherent for every $(x, y, t) \in [0, 1]^3$, that is the imprecise assessment $\mathcal{I} = [0, 1]^3$ on $\mathcal{F}$ is t-coherent. Also $\mathcal{I} = [0, 1]^3$ on $\mathcal{F}' = (C|B, B|A, C|A)$ is t-coherent.[5]

For the proof of Theorem 4 (which will be given below) we use an algorithm which computes the interval of coherent probability extensions $[z', z'']$ from a coherent interval-valued probability assessment (see [4, Algorithm 2]). For the sake of keeping the paper a bit more self-contained, we now sketch this algorithm and adapt it to deal with precise coherent probability assessments.

Algorithm 1. Let $\mathcal{F}_n = (E_1|H_1, \dots, E_n|H_n)$ be a sequence of conditional events and $\mathcal{P}_n = (p_1, \dots, p_n)$ be a coherent precise probability assessment on $\mathcal{F}_n$, where $p_j = p(E_j|H_j) \in [0, 1]$, $j = 1, \dots, n$. Moreover, let $E_{n+1}|H_{n+1}$ be a further conditional event and denote by J_{n+1} the set $\{1, \dots, n+1\}$. The steps below describe the computation of the lower bound z' (resp., the upper bound z'') for the coherent extensions $z = p(E_{n+1}|H_{n+1})$.

- **Step 0.** Expand the expression

$$\bigwedge_{j \in J_{n+1}} (E_j H_j \vee \neg E_j H_j \vee \neg H_j)$$

and denote by $C_1, \dots, C_m$ the constituents contained in $H_0 = \bigvee_{j \in J_{n+1}} H_j$. Then, construct the following system in the unknowns $\lambda_1, \dots, \lambda_m, z$

$$\begin{cases} \sum_{r: C_r \subseteq E_{n+1} H_{n+1}} \lambda_r = z \sum_{r: C_r \subseteq H_{n+1}} \lambda_r \,; \\ \sum_{r: C_r \subseteq E_j H_j} \lambda_r = p_j \sum_{r: C_r \subseteq H_j} \lambda_r, \quad j \in J_n \,; \\ \sum_{r \in J_m} \lambda_r = 1; \quad \lambda_r \geq 0, \ r \in J_m \,. \end{cases} \tag{5}$$

- **Step 1.** Check the solvability of system (5) under the condition $z = 0$ (resp., $z = 1$). If the system (5) is not solvable go to Step 2, otherwise go to Step 3.
- **Step 2.** Solve the following linear programming problem

$$\text{Compute:} \qquad \gamma' = \min \sum_{r: C_r \subseteq E_{n+1} H_{n+1}} \lambda_r$$

$$\text{(respectively:} \qquad \gamma'' = \max \sum_{r: C_r \subseteq E_{n+1} H_{n+1}} \lambda_r)$$

subject to:

$$\begin{cases} \sum_{r: C_r \subseteq E_j H_j} \lambda_r = p_j \sum_{r: C_r \subseteq H_j} \lambda_r, \ j \in J_n \,; \\ \sum_{r: C_r \subseteq H_{n+1}} \lambda_r = 1; \ \lambda_r \geq 0, \ r \in J_m \,. \end{cases}$$

[5] For proving total coherence of $\mathcal{I}$ on $\mathcal{F}$ (resp., $\mathcal{F}'$) it is sufficient to check that the assessment $\{0, 1\}^3$ on $\mathcal{F}$ (resp., $\mathcal{F}'$) is t-coherent [21, Theorem 7], i.e., each of the eight vertices of the unit cube is coherent. Coherence can be checked, for example, by applying Algorithm 1 of [21] or by the CkC-package [2].

The minimum γ' (respectively the maximum γ'') of the *objective function* coincides with z' (respectively with z'') and the procedure stops.

- **Step 3.** For each subscript $j \in J_{n+1}$, compute the maximum M_j of the function $\Phi_j = \sum_{r:C_r \subseteq H_j} \lambda_r$, subject to the constraints given by the system (5) with $z = 0$ (respectively $z = 1$). We have the following three cases:
  1. $M_{n+1} > 0$;
  2. $M_{n+1} = 0$, $M_j > 0$ for every $j \neq n+1$;
  3. $M_j = 0$ for $j \in I_0 = J \cup \{n+1\}$, with $J \neq \emptyset$.

  In the first two cases $z' = 0$ (respectively $z'' = 1$) and the procedure stops.
  In the third case, defining $I_0 = J \cup \{n+1\}$, set $J_{n+1} = I_0$ and $(\mathcal{F}_n, \mathcal{P}_n) = (\mathcal{F}_J, \mathcal{P}_J)$; then go to Step 0.

The procedure ends in a finite number of cycles by computing the value z' (respectively z'').

Theorem 3. *Let A, B, C be three logically independent events and $(x, y, t) \in [0,1]^3$ be a (coherent) assessment on the family $\big(C|B, B|A, A|A \vee B\big)$. Then, the extension $z = P(C|A)$ is coherent if and only if $z \in [z', z'']$, where*

$$[z', z''] = \begin{cases} [0,1], & t = 0; \\ \left[\max\left\{0, xy - \dfrac{(1-t)(1-x)}{t}\right\}, \min\left\{1, (1-x)(1-y) + \dfrac{x}{t}\right\}\right], & t > 0\,. \end{cases}$$

The following detailed proof of Theorem 3 is obtained by applying Algorithm 1 in a symbolic way.[6]

Proof. *Computation of the lower probability bound z' on $C|A$.*
Input. $\mathcal{F}_n = (C|B, B|A, A|A \vee B)$, $E_{n+1}|H_{n+1} = C|A$.
Step 0. The constituents associated with $(C|B, B|A, A|A \vee B, C|A)$ and contained in $\mathcal{H}_0 = A \vee B$ are $C_1 = ABC$, $C_2 = AB\neg C$, $C_3 = A\neg BC$, $C_4 = A\neg B\neg C$, $C_5 = \neg ABC$, and $C_6 = \neg AB\neg C$. We construct the following starting system with unknowns $\lambda_1, \dots, \lambda_6, z$:

$$\begin{cases} \lambda_1 + \lambda_3 = z(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), \quad \lambda_1 + \lambda_5 = x(\lambda_1 + \lambda_2 + \lambda_5 + \lambda_6), \\ \lambda_1 + \lambda_2 = y(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = t(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6), \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1, \;\; \lambda_i \geq 0, \;\; i = 1, \dots, 6\,. \end{cases} \tag{6}$$

Step 1. By setting $z = 0$ in System (6), we obtain

$$\begin{cases} \lambda_1 + \lambda_3 = 0, \\ \lambda_5 = x(\lambda_2 + \lambda_5 + \lambda_6), \\ \lambda_2 = y(\lambda_2 + \lambda_4), \\ \lambda_2 + \lambda_4 = t, \\ \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 = 1, \\ \lambda_i \geq 0, \; i = 1, \dots, 6\,; \end{cases} \iff \begin{cases} \lambda_1 = \lambda_3 = 0, \\ \lambda_2 = yt, \\ \lambda_4 = t(1-y), \\ \lambda_5 = x(yt + 1 - t), \\ \lambda_6 = (1-t)(1-x) - xyt, \\ \lambda_i \geq 0, \; i = 1, \dots, 6\,. \end{cases} \tag{7}$$

As $(x, y, t) \in [0,1]^3$, it holds that: $\lambda_2 = yt \geq 0$, $\lambda_4 = t(1-y) \geq 0$, and $\lambda_5 = x(yt + 1 - t) \geq 0$. Thus, System (7) is solvable iff $\lambda_6 \geq 0$, that is $t(1-x+xy) \leq 1-x$. We distinguish two cases: (i) $t(1-x+xy) > 1-x$; (ii) $t(1-x+xy) \leq 1-x$. In Case (i), System (7) is not solvable and we go to Step 2 of the algorithm. In Case (ii), System (7) is solvable and we go to Step 3.

[6] Alternative proofs of Theorem 3 can be obtained by applying other equivalent methods [8,10,14,45].

Case (i). By Step 2 we have the following linear programming problem:
Compute $z' = \min(\lambda_1 + \lambda_3)$ *subject to:*

$$\begin{cases} \lambda_1 + \lambda_5 = x(\lambda_1 + \lambda_2 + \lambda_5 + \lambda_6), \;\; \lambda_1 + \lambda_2 = y(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = t(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6), \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \;\; \lambda_i \geq 0, \; i = 1, \dots, 6. \end{cases} \tag{8}$$

As $t(1 - x + xy) > 1 - x \geq 0$, it holds that $t > 0$. In this case, the constraints in (8) can be rewritten in the following way

$$\begin{cases} \lambda_1 + \lambda_5 = x\big(y + \frac{1-t}{t}\big), \\ \lambda_1 + \lambda_2 = y, \\ \lambda_5 + \lambda_6 = \frac{1-t}{t}, \\ \lambda_3 + \lambda_4 = 1 - y, \\ \lambda_i \geq 0, \; i = 1, \dots, 6, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \lambda_5 = xy + x\frac{1-t}{t} - \lambda_1, \\ \lambda_2 = y - \lambda_1, \\ \lambda_6 = \frac{1-t}{t} - xy - x\frac{1-t}{t} + \lambda_1, \\ \lambda_4 = 1 - y - \lambda_3, \\ \lambda_i \geq 0, \; i = 1, \dots, 6, \end{cases}$$

that is

$$\begin{cases} \max\big\{0, xy - \frac{(1-t)(1-x)}{t}\big\} \leq \lambda_1 \leq \min\big\{y, xy + x\frac{1-t}{t}\big\}, \\ \lambda_2 = y - \lambda_1, \;\; 0 \leq \lambda_3 \leq 1 - y, \;\; \lambda_4 = 1 - y - \lambda_3, \\ \lambda_5 = xy + x\frac{1-t}{t} - \lambda_1, \;\; \lambda_6 = \frac{(1-t)(1-x)}{t} - xy + \lambda_1. \end{cases} \tag{9}$$

As $t(1 - x + xy) > 1 - x \geq 0$, it holds that $xy - (1 - x)(1 - t)/t > 0$. Thus, the minimum of $(\lambda_1 + \lambda_3)$ subject to (9) is obtained at $(\lambda_1', \lambda_3') = (xy - (1 - t)(1 - x)/t, 0)$. The *procedure stops* yielding as *output* $z' = \lambda_1' + \lambda_3' = xy - (1 - t)(1 - x)/t > 0$.
Case (ii). We take Step 3 of the algorithm. We denote by Λ and $\mathcal{S}$ the vector of unknowns $(\lambda_1, \dots, \lambda_6)$ and the set of solution of System (7), respectively. We consider the following linear functions (associated with the conditioning events $H_1 = B$, $H_2 = H_4 = A$, $H_3 = A \vee B$) and their maxima in $\mathcal{S}$:

$$\begin{aligned} &\Phi_1(\Lambda) = \textstyle\sum_{r:C_r \subseteq B} \lambda_r = \lambda_1 + \lambda_2 + \lambda_5 + \lambda_6, \\ &\Phi_2(\Lambda) = \Phi_4(\Lambda) = \textstyle\sum_{r:C_r \subseteq A} \lambda_r = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \\ &\Phi_3(\Lambda) = \textstyle\sum_{r:C_r \subseteq A \vee B} \lambda_r = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6\,, \\ &M_i = \max_{\Lambda \in \mathcal{S}} \Phi_i(\Lambda), \;\; i = 1, 2, 3, 4\,. \end{aligned} \tag{10}$$

By (7) we obtain: $\Phi_1(\Lambda) = yt + 1 - t$, $\Phi_2(\Lambda) = \Phi_4(\Lambda) = t$, and $\Phi_3(\Lambda) = 1$, $\forall \Lambda \in \mathcal{S}$. Then, $M_1 = yt + 1 - t$, $M_2 = M_4 = t$, and $M_3 = 1$. We consider two subcases: $t > 0$; $t = 0$. If $t > 0$, then $M_4 > 0$ and we are in the first case of Step 3. Thus, the *procedure stops* and yields $z' = 0$ as *output*. If $t = 0$, then $M_1 > 0, M_3 > 0$ and $M_2 = M_4 = 0$. Hence, we are in third case of Step 3 with $J = \{2\}$, $I_0 = \{2, 4\}$ and the procedure restarts with Step 0, with $\mathcal{F}_n$ replaced by $\mathcal{F}_J = (B|A)$.
(2^{nd} cycle) Step 0. The constituents associated with $(B|A, C|A)$, contained in A, are $C_1 = ABC$, $C_2 = AB\neg C$, $C_3 = A\neg BC$, $C_4 = A\neg B\neg C$. The starting system is

$$\begin{cases} \lambda_1 + \lambda_2 = y(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), \;\; \lambda_1 + \lambda_3 = z(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \;\; \lambda_i \geq 0, \;\; i = 1, \dots, 4\,. \end{cases} \tag{11}$$

(2^{nd} cycle) Step 1. By setting $z = 0$ in System (11), we obtain

$$\big\{\, \lambda_2 = y, \;\; \lambda_1 + \lambda_3 = 0, \;\; \lambda_4 = 1 - y, \;\; \lambda_i \geq 0, \;\; i = 1, \dots, 4\,. \tag{12}$$

As $y \in [0, 1]$, System (12) is always solvable; thus, we go to Step 3.

(2^{nd} cycle) Step 3. We denote by Λ and $\mathcal{S}$ the vector of unknowns $(\lambda_1, \dots, \lambda_4)$ and the set of solution of System (12), respectively. The conditioning events are $H_2 = A$ and $H_4 = A$; then the associated linear functions are: $\Phi_2(\Lambda) = \Phi_4(\Lambda) = \sum_{r: C_r \subseteq A} \lambda_r = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$. From System (12), we obtain: $\Phi_2(\Lambda) = \Phi_4(\Lambda) = 1, \;\; \forall \Lambda \in \mathcal{S}$; so that $M_2 = M_4 = 1$. We are in the first case of Step 3 of the algorithm; then the *procedure stops* and yields $z' = 0$ as *output*.

To summarize, for any $(x, y, t) \in [0, 1]^3$ on $(C|B, B|A, A|A \vee B)$, we have computed the coherent lower bound z' on $C|A$. In particular, if $t = 0$, then $z' = 0$. Moreover, if $t > 0$ and $t(1 - x + xy) \le 1 - x$, that is $xy - (1-t)(1-x)/t \le 0$, we also have $z' = 0$. Finally, if $t(1-x+xy) > 1-x$, then $z' = xy - (1 - t)(1 - x)/t$.

Computation of the upper probability bound z'' on $C|A$

Input and *Step 0* are the same as in the proof of z'.

Step 1. By setting $z = 1$ in System (6), we obtain

$$\begin{cases} \lambda_2 + \lambda_4 = 0, \\ \lambda_1 + \lambda_5 = x(\lambda_1 + \lambda_5 + \lambda_6), \\ \lambda_1 = y(\lambda_1 + \lambda_3), \;\; \lambda_1 + \lambda_3 = t, \\ \lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 = 1, \\ \lambda_i \ge 0, \; i = 1, \dots, 6\,, \end{cases} \iff \begin{cases} \lambda_1 = yt, \;\; \lambda_3 = t(1 - y), \\ \lambda_2 = \lambda_4 = 0, \\ \lambda_5 = x - xt + xyt - yt, \\ \lambda_6 = (1 - x)[1 - t(1 - y)], \\ \lambda_i \ge 0, \; i = 1, \dots, 6\,. \end{cases} \tag{13}$$

As $(x, y, t) \in [0, 1]^3$, it holds that: $\lambda_1 = yt \ge 0$, $\lambda_3 = t(1 - y) \ge 0$, and $\lambda_6 = (1 - x)[1 - t(1 - y)] \ge 0$. Then, System (13) is solvable if and only if $\lambda_5 \ge 0$, i.e., $t(x + y - xy) \le x$. We distinguish two cases: (i) $x + xyt - xt - yt < 0$; (ii) $x + xyt - xt - yt \ge 0$. In Case (i), System (13) is not solvable and we go to Step 2 of the algorithm. In Case (ii), System (13) is solvable and we go to Step 3.

Case (i). We take Step 2 and consider the following linear programming problem: *Compute* $z'' = \max(\lambda_1 + \lambda_3)$, *subject to the constraints in* (8). As $x + xyt - xt - yt < 0$, that is $t(x + y - xy) > x \ge 0$, it holds that $t > 0$. In this case, the constraints in (8) can be rewritten as in (9). Since $x + xyt - xt - yt < 0$, it holds that $x + xyt - xt < yt \le y$. Thus, we obtain the maximum of $(\lambda_1 + \lambda_3)$ subject to (9) at $(\lambda_1'', \lambda_3'') = (xy - x + x/t, 1 - y)$. The *procedure stops* and yields the following *output*: $z'' = 1 - y - x + xy + x/t = (1 - x)(1 - y) + x/t$.

Case (ii). We take Step 3 of the algorithm. We denote by Λ and $\mathcal{S}$ the vector of unknowns $(\lambda_1, \dots, \lambda_6)$ and the set of solution of System (13), respectively. We consider the functions given in (10). From System (13), we obtain $M_1 = yt + 1 - t$, $M_2 = M_4 = t$, and $M_3 = 1$. If $t > 0$, then $M_4 > 0$ and we are in the first case of Step 3. Thus, the *procedure stops* and yields $z'' = 1$ as *output*. If $t = 0$, then $M_1 > 0, M_3 > 0$ and $M_2 = M_4 = 0$. Hence, we are in the third case of Step 3 with $J = \{2\}$, $I_0 = \{2, 4\}$ and the procedure restarts with Step 0, with $\mathcal{F}_n$ replaced by $\mathcal{F}_J = (E_2|H_2) = (B|A)$ and $\mathcal{A}_n$ replaced by $\mathcal{A}_J = ([\alpha_2, \beta_2]) = ([y, y])$.

(2^{nd} cycle) Step 0. This is the same as the (2^{nd} cycle) Step 0 in the proof of z'.

(2^{nd} cycle) Step 1. By setting $z = 1$ in System (6), we obtain

$$\left\{ \lambda_1 = y, \;\; \lambda_3 = 1 - y, \;\; \lambda_2 + \lambda_4 = 0, \;\; \lambda_i \ge 0, \;\; i = 1, \dots, 4\,. \right. \tag{14}$$

As $y \in [0, 1]$, System (14) is always solvable; thus, we go to Step 3.

(2^{nd} cycle) Step 3. Like in the (2^{nd} cycle) Step 3 of the proof of z', we obtain $M_4 = 1$. Thus, the procedure stops and yields $z'' = 1$ as output.

To summarize, for any assessment $(x, y, t) \in [0, 1]^3$ on $(C|B, B|A, A|A \vee B)$, we have computed the coherent upper probability bound z'' on $C|A$. In particular, if $t = 0$, then $z'' = 1$. Moreover, if $t > 0$ and $t(x + y - xy) \le x$, that is $(x + y - xy) \le \frac{x}{t} \iff \frac{x}{t} - x - y + xy \ge 0 \iff (1 - x)(1 - y) + \frac{x}{t} \ge 1$, we also have $z'' = 1$. Finally, if $t(x + y - xy) > x$, then $z'' = (1 - x)(1 - y) + \frac{x}{t}$. $\square$

Theorem 4. *Let A, B, C be three logically independent events and $\mathcal{I} = ([x_1, x_2] \times [y_1, y_2] \times [t_1, t_2]) \subseteq [0,1]^3$ be an imprecise (totally-coherent) assessment on $\big(C|B, B|A, A|A \vee B\big)$. Then, the set Σ of the coherent extension of $\mathcal{I}$ is the interval $[z^*, z^{**}]$, where $[z^*, z^{**}] =$*

$$\begin{cases} [0,1], & t = 0; \\ \left[\max\left\{0, x_1 y_1 - \dfrac{(1-t_1)(1-x_1)}{t_1}\right\}, \min\left\{1, (1-x_2)(1-y_1) + \dfrac{x_2}{t_1}\right\}\right], & t > 0\,. \end{cases}$$

Proof. We observe that $\Sigma = \bigcup_{\mathcal{P} \in \mathcal{I}} [z'_{\mathcal{P}}, z''_{\mathcal{P}}] = [z^*, z^{**}]$. If $t_1 = 0$, we obtain $[z^*, z^{**}] = [0,1]$ by Theorem 3. If $t_1 > 0$, the proof is straightforward by observing that the lower bound z' in Theorem 3 is non-decreasing in the arguments x, y, t; moreover, the upper bound z'' is non-decreasing in the argument x, while it is non-increasing in the argument y and t. $\square$

Remark 5. By applying Theorem 4 with $x_1 = y_1 = 1 - \varepsilon$, $t_1 > 0$, and $x_2 = y_2 = t_2 = 1$ we obtain $z^* = \max\left\{0, (1-\varepsilon)^2 - \frac{(1-\varepsilon)\varepsilon}{t_1}\right\}$ and $z^{**} = 1$, with $z^* = 0$ if and only if $\varepsilon = 1$ or $(\varepsilon < 1) \wedge (t_1 \leq \varepsilon/(1-\varepsilon))$.

5. Weak Transitivity involving (negated) defaults

Let A, B, C be three logically independent events. By Remark 4, the p-consistent knowledge base $(B \mathrel{|\!\sim} C, A \mathrel{|\!\sim} B)$ neither p-entails $A \mathrel{|\!\sim} C$ nor p-entails $A \mathrel{|\!\not\sim} C$. This will be denoted by $(B \mathrel{|\!\sim} C, A \mathrel{|\!\sim} B) \not\models_p A \mathrel{|\!\sim} C$ and $(B \mathrel{|\!\sim} C, A \mathrel{|\!\sim} B) \not\models_p A \mathrel{|\!\not\sim} C$, respectively.

Theorem 5. $(B \mathrel{|\!\sim} C, A \mathrel{|\!\sim} B, A \vee B \mathrel{|\!\not\sim} \neg A) \models_p A \mathrel{|\!\sim} C$.

Proof. By Remark 4, the knowledge base $\mathcal{K} = (B \mathrel{|\!\sim} C, A \mathrel{|\!\sim} B, A \vee B \mathrel{|\!\not\sim} \neg A)$ is p-consistent. Based on Remark 3, we set $\mathcal{I}_{\mathcal{K}} = \{1\} \times \{1\} \times]0,1]$ and $\mathcal{F}_{\mathcal{K}} = \big(C|B, B|A, A|A \vee B\big)$. Let $\mathcal{P}$ be any precise coherent assessment on $\mathcal{F}_{\mathcal{K}}$ such that $\mathcal{P} \in \mathcal{I}_{\mathcal{K}}$, i.e., $\mathcal{P} = (1, 1, t)$, with $t \in]0,1]$. From Theorem 3, the interval of coherent extensions from $\mathcal{P}$ on $\mathcal{F}_{\mathcal{K}}$ to $C|A$ is $[z'_{\mathcal{P}}, z''_{\mathcal{P}}] = [1,1]$. Then, by Equation (1), the set of coherent extensions to $C|A$ from $\mathcal{I}_{\mathcal{K}}$ on $\mathcal{F}_{\mathcal{K}}$ is $\bigcup_{\mathcal{P} \in \mathcal{I}_{\mathcal{K}}} [z'_{\mathcal{P}}, z''_{\mathcal{P}}] = [1,1]$. $\square$

The inference rules expressed in Theorem 5 and in Theorem 7 have been studied within a different semantics by [19].

By Remark 4, we also observe that the p-consistent knowledge base $(B \mathrel{|\!\sim} C, A \mathrel{|\!\not\sim} \neg B)$ does not p-entail $A \mathrel{|\!\not\sim} \neg C$.

Theorem 6. $(B \mathrel{|\!\sim} C, A \mathrel{|\!\not\sim} \neg B, A \vee B \mathrel{|\!\not\sim} \neg A) \models_p A \mathrel{|\!\not\sim} \neg C$.

Proof. By Remark 4, the knowledge base $\mathcal{K} = (B \mathrel{|\!\sim} C, A \mathrel{|\!\not\sim} \neg B, A \vee B \mathrel{|\!\not\sim} \neg A)$ is p-consistent. Based on Remark 3, we set $\mathcal{I}_{\mathcal{K}} = \{1\} \times]0,1] \times]0,1]$ and $\mathcal{F}_{\mathcal{K}} = \big(C|B, B|A, A|A \vee B\big)$. Let $\mathcal{P}$ be any precise coherent assessment on $\mathcal{F}_{\mathcal{K}}$ such that $\mathcal{P} \in \mathcal{I}_{\mathcal{K}}$, i.e., $\mathcal{P} = (1, y, t)$, with $y \in]0,1]$ and $t \in]0,1]$. From Theorem 3, the interval of coherent extensions from $\mathcal{P}$ on $\mathcal{F}_{\mathcal{K}}$ to $C|A$ is $[z'_{\mathcal{P}}, z''_{\mathcal{P}}] = [y,1]$. Then, by Equation (1), the set of coherent extensions to $C|A$ from $\mathcal{I}_{\mathcal{K}}$ on $\mathcal{F}_{\mathcal{K}}$ is $\bigcup_{\mathcal{P} \in \mathcal{I}_{\mathcal{K}}} [z'_{\mathcal{P}}, z''_{\mathcal{P}}] = \bigcup_{(y,t) \in]0,1] \times]0,1]} [y, 1] =]0,1]$. Therefore, the set of coherent extensions on $\neg C|A$ is $[0,1[$. $\square$

Remark 6. We observe that

$$(B \mathrel{|\!\sim} C, A \mathrel{|\!\not\sim} \neg B, A \vee B \mathrel{|\!\not\sim} \neg A) \models_p AB \mathrel{|\!\sim} C\,, \tag{15}$$

and hence

$$(B \mid\!\sim C, A \mid\!\sim B, A \vee B \not\mid\!\sim \neg A) \models_p AB \mid\!\sim C\,, \tag{16}$$

because $p(C|A \wedge B) = 1$ follows from the probabilistic constraints $p(C|B) = 1$, $p(B|A) > 0$, and $p(A|A \vee B) > 0$. Indeed, these constraints imply $p(\neg C|B) = 1 - P(C|B) = 0$ and $p(AB|A \vee B) = p(B|A)p(A|A \vee B) > 0$; then, by the probability compound theorem

$$p(ABC|A \vee B) = p(C|AB)p(AB|A \vee B)$$

it follows that

$$p(C|AB) = \frac{p(ABC|A\vee B)}{p(AB|A\vee B)} = \frac{p(ABC|A\vee B)}{p(ABC|A\vee B)+p(AB\neg C|A\vee B)} =$$
$$= \frac{p(ABC|A\vee B)}{p(ABC|A\vee B)+p(A|\neg CB)p(\neg C|B)p(B|A\vee B)} = \frac{p(ABC|A\vee B)}{p(ABC|A\vee B)} = 1\,.$$

We recall the probabilistic Cut Rule given in [20]:

$$\text{if } p(C|AB) = x \text{ and } p(B|A) = y, \text{ then } p(C|A) = z \in [xy, xy + 1 - y]\,.$$

In particular, for $x = 1$ and for any $y > 0$ it follows that $z > 0$; moreover, for $x = 1$ and $y = 1$ it follows that $z = 1$. This means in terms of defaults:

$$AB \mid\!\sim C, A \not\mid\!\sim \neg B \models_p A \not\mid\!\sim \neg C\,; \tag{17}$$

$$AB \mid\!\sim C, A \mid\!\sim B \models_p A \mid\!\sim C\,. \tag{18}$$

We also observe that the assessment $p(C|B) = p(C|AB) = p(B|A) = 1$ is coherent, then from (18) we obtain

$$B \mid\!\sim C, A \mid\!\sim B, AB \mid\!\sim C \models_p A \mid\!\sim C \tag{19}$$

which is a (weaker) version of Theorem 5 where the premise $A \vee B \not\mid\!\sim \neg A$ has been replaced by $AB \mid\!\sim C$. Likewise, from (17) we obtain

$$B \mid\!\sim C, A \not\mid\!\sim \neg B, AB \mid\!\sim C \models_p A \not\mid\!\sim \neg C \tag{20}$$

which is a (weaker) version of Theorem 6.

Theorem 7. *$(B \mid\!\sim C, A \mid\!\sim B, B \not\mid\!\sim \neg A) \models_p A \mid\!\sim C$.*

Proof. It can be shown that the assessment $[0,1]^3$ on $(C|B, B|A, A|B)$ is t-coherent. Then, $\mathcal{K} = (B \mid\!\sim C, A \mid\!\sim B, B \not\mid\!\sim \neg A)$ is p-consistent. We set $\mathcal{I}_{\mathcal{K}} = \{1\} \times \{1\} \times]0,1]$ and $\mathcal{F}_{\mathcal{K}} = \big(C|B, B|A, A|B\big)$. We observe that $A|B \subseteq A|A \vee B$, where the binary relation $\subseteq$ denotes the well-known Goodman and Nguyen inclusion relation between conditional events (see, e.g., [23,27]). Coherence requires that $p(A|B) \leq p(A|A \vee B)$. Let $\mathcal{P}$ be any precise coherent assessment on $\mathcal{F}_{\mathcal{K}}$ such that $\mathcal{P} \in \mathcal{I}_{\mathcal{K}}$, i.e., $\mathcal{P} = (1, 1, w)$, with $w \in]0,1]$. Thus, for any coherent extension $\mathcal{P}' = (1, 1, w, t)$ of $\mathcal{P}$ on $(\mathcal{F}_{\mathcal{K}}, A|A \vee B)$, it holds that $0 < w \leq t$. Then, $\mathcal{K}' = (B \mid\!\sim C, A \mid\!\sim B, B \not\mid\!\sim \neg A, A \vee B \not\mid\!\sim \neg A)$ is p-consistent. Thus, by Theorem 5, $\mathcal{K}' \models_p A \mid\!\sim C$. Then, for every coherent extension $\mathcal{P}'' = (1, 1, w, t, z)$ of $\mathcal{P}'$ on $(\mathcal{F}_{\mathcal{K}'}, C|A)$ it holds that $z = 1$. By *reductio ad absurdum*, if for some $z < 1$ the extension $(1, 1, w, z)$ on $(\mathcal{F}_{\mathcal{K}}, C|A)$ of $\mathcal{P} \in \mathcal{I}_{\mathcal{K}}$ on $\mathcal{F}_{\mathcal{K}}$ were coherent, then—with $0 < w \leq t$ and $z < 1$—the assessment $(1, 1, w, t, z)$ on $(\mathcal{F}_{\mathcal{K}'}, C|A)$ would be coherent, which contradicts the conclusion $z = 1$ above. Thus, for every coherent extension $(1, 1, w, z)$ of $\mathcal{P} \in \mathcal{I}_{\mathcal{K}}$ on $(\mathcal{F}_{\mathcal{K}}, C|A)$ it holds that $z = 1$. $\square$

Theorem 8. $(B \mid\!\sim C, A \mid\!\not\sim \neg B, B \mid\!\not\sim \neg A) \models_p A \mid\!\not\sim \neg C$.

Proof. The proof exploits Theorem 6 and is similar to the proof of Theorem 7. □

Remark 7. Of course by Definition 8, Theorem 5 to Theorem 8 can be rewritten in terms of probability constraints. Theorem 5, for example, would then read as follows: $p(C|B) = 1$, $p(B|A) = 1$, and $p(A|A \vee B) > 0$ implies $p(C|A) = 1$.

6. Applications to classical categorical syllogisms

Classical categorical syllogisms are arguments consisting of two premises and a conclusion. Theorem 3 can be exploited to construct a coherence-based probability semantics of classical categorical syllogisms, specifically those of the well-known syllogistic *Figure 1*, which were already investigated by Aristotle in his "Analytica Priora." *Figure 1* syllogisms are valid transitive argument forms which are composed of universally/existentially quantified statements and their respective negated versions (see, e.g., [34]). Examples of valid syllogisms of *Figure 1* are Modus Barbara (*Every M is P, Every S is M, therefore Every S is P*) and Modus Darii (*Every M is P, Some S is M, therefore Some S is P*).

There are four basic syllogistic sentence types involved in the construction of the syllogisms: (A) *Every a is b*, (E) *No a is b*, (I) *Some a is b*, and (O) *Some a is not b*, where "a" and "b" denote two of the three categorical terms M, P, or S. We observe that from a first order logic point of view, the S, M, and P terms involved in the basic syllogistic sentence types are usually interpreted as predicates, which are interpreted in our probabilistic semantics by events. Indeed, we relate each predicate to an event as follows. Imagine a random experiment where the (random) outcome is denoted by X. Consider, for example, the predicate S. Depending on the result of the experiment, X may satisfy or not satisfy the predicate S. Then, we denote by E_S the event "X satisfies S" (the event E_S is true if X satisfies the predicate S and E_S is false if X does not satisfy S). We conceive the predicate S as the event E_S, which will be true or false. Thus, we simply identify E_S by S (in this sense S is both a predicate and an event). The same reasoning applies to the syllogistic P and M terms, which are in our context both predicates and events.

The basic syllogistic sentence types (A) *Every a is b*, (E) *No a is b*, (I) *Some a is b*, and (O) *Some a is not b* can be interpreted by (A) $a \mid\!\sim b$, (E) $a \mid\!\sim \neg b$, (I) $a \mid\!\not\sim \neg b$, and (O) $a \mid\!\not\sim b$, respectively. Table 1 presents the respective probabilistic interpretation (see also [11]).

Based on this interpretation of the basic syllogistic sentence types, we construct default versions of classical categorical syllogisms. The Weak Transitivity rule in the statement of Theorem 5, for example, is our default version of Modus Barbara, i.e., $(M \mid\!\sim P, S \mid\!\sim M$, and $S \vee M \mid\!\not\sim \neg S) \models_p S \mid\!\sim P$. By weakening the conclusion of Modus Barbara (see Remark 2), we obtain the default version of Modus Barbari, i.e., $(M \mid\!\sim P, S \mid\!\sim M$, and $S \vee M \mid\!\not\sim \neg S) \models_p S \mid\!\not\sim \neg P$. Theorem 6 is our default version of Modus Darii, i.e., $(M \mid\!\sim P, S \mid\!\not\sim \neg M$, and $S \vee M \mid\!\not\sim \neg S) \models_p S \mid\!\not\sim \neg P$. We observe that in our approach the premise $S \vee M \mid\!\not\sim \neg S$, that we call EI_1, can serve as an *existential import* assumption for the validity of the *Figure 1* syllogisms. In case of Modus Barbara, for example, the (major and minor) premises alone $(M \mid\!\sim P, S \mid\!\sim M)$ do not p-entail the conclusion $S \mid\!\sim P$ (see Section 5, see also Remark 4 for the probabilistic version). Similarly, the (major and minor) premises alone $(M \mid\!\sim P, S \mid\!\not\sim \neg M)$ do not p-entail the conclusion $S \mid\!\not\sim \neg P$ in the case of Modus Darii. Thus, by adding the existential import assumption $S \vee M \mid\!\not\sim \neg S$ to the respective premise sets, the validity of Modus Barbara and Modus Darii is guaranteed.

In terms of probabilistic constraints the existential import assumption EI_1 is expressed by $p(S|S \vee M) > 0$; moreover, Theorems 5 and 6 are expressed, respectively, by

$$\text{(Modus Barbara)} \quad p(P|M) = 1, p(M|S) = 1, \text{ and } EI_1 \implies p(P|S) = 1\,,$$

$$\text{(Modus Darii)} \quad p(P|M) = 1, p(M|S) > 0, \text{ and } EI_1 \implies p(P|S) > 0\,.$$

However, by instantiating S, M, P in Theorem 3, we obtain that the whole interval $[0, 1]$ on $P|S$ is a coherent extension of the probabilistic constraints $p(P|M) = 1$, $p(M|S) = 1$ (or $p(M|S) > 0$), and $p(S|S \vee M) = 0$. Then, ($p(P|M) = 1$, $p(M|S) = 1$, and $p(S|S \vee M) = 0$) does not imply $p(P|S) = 1$. In other words, Modus Barbara (and hence Transitivity) is not valid when we replace in the premises set the existential import assumption EI_1 by its negation (i.e., $p(S|S \vee M) = 0$). Similarly for Modus Darii: ($p(P|M) = 1$, $p(M|S) > 0$, and $p(S|S \vee M) = 0$) does not imply $p(P|S) > 0$. As noted above, Modus Barbara also implies Modus Barbari:

$$\text{(Modus Barbari)} \quad p(P|M) = 1, p(M|S) = 1, \text{ and } EI_1 \implies p(P|S) > 0\,.$$

We note that in our approach we need an existential import assumption for all: Modus Barbara, Modus Barbari, and Modus Darii. From a first order logic point of view, however, for both Modus Barbara and Modus Darii the existential import assumption is not required.[7] Historically, it seems plausible to us that an existential import has been assumed as an (at least implicit) background assumption in classical categorical syllogisms.

The considered *Figure 1* syllogisms can also be expressed with the (stronger) notion of existential import[8] $EI_2 : p(S|M) > 0$ (see [17]). In particular, Modus Barbara and Modus Darii are presented in terms of defaults in Theorem 7 and in Theorem 8, respectively. Moreover, in all previous versions of syllogisms we do not presuppose any positive antecedent probabilities in our framework. Assuming the positive antecedent probability $EI_3 : p(S) > 0$ would be yet another existential import assumption sufficient for the validity of Modus Barbara, Modus Barbari, and Modus Darii which is stronger than EI_1. Indeed, coherence requires that

$$p(S) = p(S \wedge (S \vee M)) = p(S|S \vee M)P(S \vee M)\,.$$

Hence, $p(S) > 0$ implies $p(S|S \vee M) > 0$ (or equivalently: $p(S|S \vee M) = 0$ implies $p(S) = 0$). Therefore, $p(S) > 0$ is stronger than $p(S|S \vee M) > 0$. Moreover, in Modus Barbara, Modus Barbari, and Modus Darii with EI_3 coherence also requires positive probability for the antecedent of the major premise, indeed, as $p(M|S) > 0$ and $p(S) > 0$, it holds that: $p(M) \geq p(M \wedge S) = p(M|S)p(S) > 0$. However, it can be proved that $p(P|M) = 1$, $p(M|S) > 0$, and $p(S) = 0$ does not imply $p(M) = 0$. The deepening of these aspects could be related to the general problem of *zero layers* largely studied in [14].

Based on Remark 6, we observe that *Figure 1* syllogisms can also be expressed with the (weaker) notion of existential import $EI_4 : p(P|MS) = 1$. In particular, Modus Barbara and Modus Darii are presented in terms of defaults in formulas (19) and (20), respectively. We note that by adding EI_4 to the premise set of the considered *Figure 1* syllogisms, we have $p(P|MS) = p(P|M) = 1$, which is postulated (in terms of conditional independence assumptions) in [11] for obtaining the validity of the corresponding syllogisms. However, in contrast to [11], we obtain the validity of these syllogisms with EI_4 even without presupposing positive antecedent probabilities.

[7] From a first order logic point of view, the existential import is trivially satisfied in Modus Darii (i.e., $\forall x(Mx \supset Px)$ and $\exists x(Sx \wedge Mx)$ logically implies $\exists x(Sx \wedge Px)$, where "$\supset$" denotes the material conditional) as the minor premise ($\exists x(Sx \wedge Mx)$) logically entails the existential import ($\exists x Sx$). Moreover, the first order logic version of Modus Barbara, (i.e., $\forall x(Mx \supset Px)$ and $\forall x(Sx \supset Mx)$ logically implies $\forall x(Sx \supset Px)$) can be validated without an existential import assumption. Indeed, Modus Barbara would also be valid, e.g., if the minor premise is vacuously true (i.e., if $\neg\exists x Sx$). For showing the logical validity of Modus Barbari (i.e., $\forall x(Mx \supset Px)$ and $\forall x(Sx \supset Mx)$ logically implies $\exists x(Sx \wedge Px)$), however, the existential import assumption ($\exists x Sx$) is required, as otherwise both premises could be (vacuously) true (i.e., $\forall x(Mx \supset Px)$ and $\forall x(Sx \supset Mx)$) while the conclusion ($\exists x(Sx \wedge Px)$) would then be false and therefore Modus Barbari would not be valid.

[8] As observed in the proof of Theorem 7, $S|M \subseteq S|S \vee M$ and by coherence it follows that $p(S|M) \leq p(S|S \vee M)$. Hence, $p(S|M) > 0$ implies $p(S|S \vee M) > 0$ but not vice versa. In this sense, EI_2 is stronger than EI_1. However, the vice versa holds in the light of the considered premises: if $p(M|S) > 0$ (i.e., the minor premise of the syllogism) and $p(S|S \vee M) > 0$, then $p(S|M) > 0$. Indeed, $p(M|S) > 0$ and $p(S|S \vee M) > 0$ implies $p(M \wedge S|S \vee M) = p(M|S)p(S|S \vee M) > 0$; moreover, as $(M \wedge S|M \vee S) \subseteq (S|M)$, it follows that $p(S|M) \geq p(M \wedge S|M \vee S) > 0$.

We are currently working on a coherence-based probability semantics for classical categorical syllogisms, where we further exploit the ideas presented above.

7. Default square of opposition

In the context of categorical syllogisms, the well-known *traditional square of opposition* is used to study logical relations among the four basic syllogistic sentence types A, E, I, and O (see, e.g., [33]), which are: contradiction, contrariety, subcontrariety, and subalternation. In this section we introduce a new interpretation of the traditional square of opposition in terms of defaults and negated defaults. We now use the notions of p-consistency (Definition 9) and p-entailment (Definition 10) to define suitable interpretations of the four logical relations among A, E, I, and O, which were defined in terms of defaults and negated defaults in Section 6 (see also Table 1).

Let d denote a sentence expressing a default or a negated default.

Definition 11 *(Contrariety).* Given two statements d_1 and d_2, we say that d_1 and d_2 are *contraries* iff the sequence (d_1, d_2) is not p-consistent.[9]

Definition 12 *(Subcontrariety).* Given two sentences d_1 and d_2, we say that d_1 and d_2 are *subcontraries* iff the sequence $(\neg d_1, \neg d_2)$ is not p-consistent.

Definition 13 *(Contradiction).* Given two sentences d_1 and d_2, we say that d_1 and d_2 are *contradictories* iff they are contraries and subcontraries.

Definition 14 *(Subalternation).* Given two sentences d_1 and d_2, we say that d_2 is a *subaltern* of d_1 iff d_1 p-entails d_2.

By coherence, it is easy to verify the following relations among the basic syllogistic sentence types A, E, I, and O:

(i) $S \mid\!\sim P$ and $S \mid\!\sim \neg P$ are contraries;
(ii) $S \not\mid\!\sim \neg P$ and $S \not\mid\!\sim P$ are subcontraries;
(iii) $S \mid\!\sim P$ and $S \not\mid\!\sim P$ are contradictories;
$S \mid\!\sim \neg P$ and $S \not\mid\!\sim \neg P$ are contradictories;
(iv) $S \not\mid\!\sim \neg P$ is a subaltern of $S \mid\!\sim P$;
$S \not\mid\!\sim P$ is a subaltern of $S \mid\!\sim \neg P$.

Based on the relations (i)–(iv) we construct a square of opposition in terms of defaults and negated defaults, which is depicted in Fig. 4. We note that in our *default square of opposition* we implicitly assume that the antecedent S must not be a self-contradictory event ($S \neq \bot$). In general, this can be interpreted as a (logical) existential import assumption, which is always presupposed in coherence-based probability logic. In our context, self-contradictory antecedents do not make any sense since the conditional event $P|S$ is *undefined* if $S \equiv \bot$.

[9] Traditionally if two statements s_1 and s_2 are contraries, then s_1 and s_2 cannot both be true. Some definitions of contrariety additionally require that "s_1 and s_2 can both be false" (for a discussion see, e.g., [29,44]). We omit the respective (probabilistic) version of this additional requirement in our definition of contrariety. Similarly, *mutatis mutandis*, in our definition of subcontrariety.

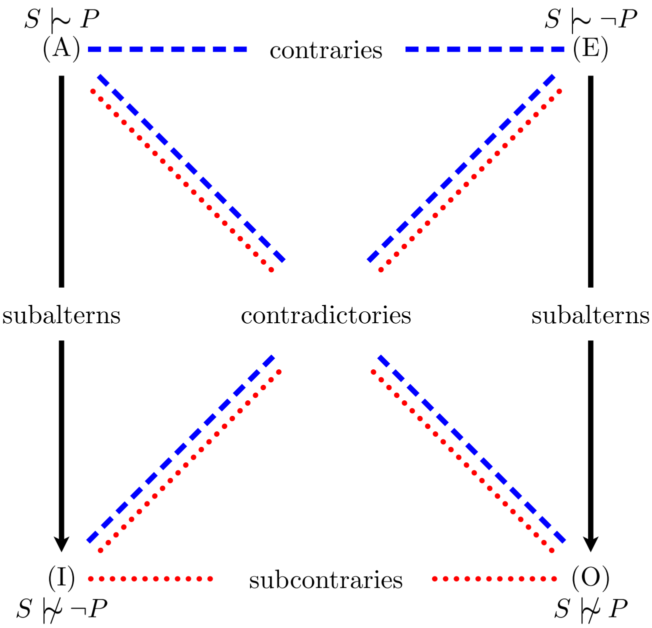

Fig. 4. Default square of opposition defined on the four sentence types introduced in Table 1. It provides a new interpretation of the traditional square of opposition (see, e.g., [33]), where the corners are labeled by "Every S is P" (A), "No S is P" (E), "Some S is P" (I), and "Some S is not P" (O).

8. Concluding remarks

In this paper we proved coherent probability propagation rules for Weak Transitivity. We applied our results to demonstrate the validity of selected inference patterns involving defaults and—new probabilistic notions of—negated defaults in the context of nonmonotonic reasoning. Moreover, we illustrated how our results can also be applied to develop a coherence-based probability semantics of classical categorical syllogisms and to construct a new version of the square of opposition.

Our definition of negated defaults, based on imprecise probabilities (Section 3), can be seen as an instance of the *wide-scope* reading of the negation of a conditional. It offers an interesting alternative to the *narrow-scope* reading, where a conditional is negated by negating its consequent [35].

Finally, we note that most of our results concerning the probability propagation rules of Weak Transitivity would also hold within standard approaches to probability where conditional probability $p(E|H)$ is defined by the ratio $p(E \wedge H)/p(H)$ (requiring positive probability of the conditioning event, $p(H) > 0$). However, in our coherence-based approach, our results *even* hold when conditioning events have zero probability. Furthermore, we observe that, by Theorem 3, $p(C|B) = 1$, $p(B|A) = 1$, and $p(A|A \vee B) = 0$ implies $0 \leq p(C|A) \leq 1$. This observation cannot be made in standard approaches to probability, as $p(A|A \vee B) = 0$ implies that the probability of the conditioning event A equals to zero, i.e., $P(A) = 0$.

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